The Convolution Theorem

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In the following, we define the Fourier transform and its inverse as

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \tag{1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx}dx. \tag{2}$$

The convolution theorem is

$$(f * g)(x) = \int_{\text{all } u} f(u)g(x-u)du. \tag{3}$$

Questions

1. Show that the Fourier transform of the delta function $f(x) = \delta(x-a)$ is $F(k) = \frac{1}{\sqrt{2\pi}}\exp(-ika)$.

2. Derive an expression for the Fourier transform $G(k)$ of the function $g(x) = \frac{1}{2}(\delta(x+a) - \delta(x-a))$, writing your answer in its simplest form.
   Find an expression for the real an imaginary parts of $G(k)$.

3. Calculate the convolution of

$$f(x) = \begin{cases} 1 & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   by carrying out the following steps

   (a) Make sketches of $f(x)$ and $g(x)$.

   (b) Sketch $g(x-u)$ vs u and $f(u)$ vs u on the same axes for $x = -3$.

   (c) Hence, identify the five ranges of x over which the convolution integral has to be evaluated.

   (d) Calculate $(f * g)(x)$ in each of these five cases.

    (e) Sketch $(f * g)(x)$.

4. Calculate the convolution of

$$f(x) = g(x) = \begin{cases} 1 & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

  by carrying out the following steps

    (a) Make sketches of $f(x)$ and $f(u)$.

    (b) Sketch $g(x - u)$ vs u and $f(u)$ vs u on the same axes for $x = -3$.

    (c) Hence, identify the four ranges of x over which the convolution integral has to be evaluated.

    (d) Calculate $(f * g)(x)$ in each of these cases.

    (e) Sketch $(f * g)(x)$.

5. Calculate the convolution of

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} e^{-2x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

  by carrying out the following steps

    (a) Make sketches of $f(x)$ and $f(u)$.

    (b) Sketch $g(x - u)$ vs u and $f(u)$ vs u on the same axes for $x = -3$.

    (c) Hence, identify the four ranges of x over which the convolution integral has to be evaluated.

    (d) Calculate $(f * g)(x)$ in each of these cases.

    (e) Sketch $(f * g)(x)$.

Solutions