Fourier Series

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For the following questions, we will define Fourier series as

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

1. Find the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Solution: We begin with a_0:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi}(1)dx \\ &= 1. \end{aligned}$$

Now, for a_n

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \cos(n\pi x/L)dx \\ &= \frac{1}{\pi}\frac{L}{n\pi}\left[\sin(n\pi x/L)\right]_0^{\pi} \\ &= \frac{1}{n\pi}(\sin n\pi - \sin 0) \\ &= 0. \end{aligned}$$

Finally, b_n

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi} \sin(n\pi x/L)dx \\ &= -\frac{1}{n\pi}\left[\cos(nx)\right]_0^{\pi} \\ &= -\frac{1}{n\pi}(\cos n\pi - 1) \end{aligned}$$

which means that for even n, $b_n = 0$ and for odd n, $b_n = 2/(n\pi)$. The series is then defined as

$$a_n = \begin{cases} 0, \text{for } n \neq 0 \\ 1, \text{for } n = 0 \end{cases} \qquad b_n = \begin{cases} 0, \text{for even } n \\ \frac{2}{n\pi}, \text{for odd } n \end{cases}$$

The first few terms are

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$$

2. Find the Fourier series of

$$f(x) = x^2, -\pi < x < \pi.$$

Then, set $x = \pi$ and show that [1]

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: We begin by noting that $f(x)$ is even, so $b_n = 0$ $\forall n$. We can find a_0 as

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{3\pi}\left[x^3\right]_{-\pi}^{\pi} \\ &= \frac{2\pi^2}{3}. \end{aligned}$$

Now, for a_n we must evaluate

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x^2 \cos nx$$

This can be integrated by parts $\int u dv = uv - \int v du$, choosing

$$u = x^2 \qquad dv = \cos nx$$
$$du = 2x \qquad v = \frac{1}{n}\sin nx$$

giving

$$a_n = \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx\Big|_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx dx\right].$$

[1]This is the Riemann Zeta function $\zeta(2)$.

The boundary term is zero as $\sin(\pm n\pi) = 0$ and the second integral can be evaluated by parts again, choosing

$$u = x \qquad dv = \sin nx$$

$$du = x \qquad v = \frac{-1}{n}\cos nx$$

giving

$$a_n = -\frac{2}{n\pi}\left[-\frac{x}{n}\cos nx\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\right].$$

The integral is zero as the limits are symmetric and $\cos(x)$ is an even function. Putting the limits into the boundary term, we have

$$\begin{aligned} a_n &= \frac{2}{n^2\pi}(\pi\cos(n\pi) - \pi\cos(-n\pi)) \\ &= \frac{4}{n^2\pi}\pi\cos n\pi \\ &= \frac{4}{n^2}(-1)^n. \end{aligned}$$

So, the Fourier series is define via

$$a_0 = \frac{2\pi^2}{3} \qquad a_n = \frac{4}{n^2}(-1)^n \qquad b_n = 0.$$

If we set $x = \pi$, we find that

$$\begin{aligned} x^2 &= \frac{2\pi^2}{6} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \\ \pi^2 &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} \end{aligned}$$

we note that $(-1)^{2n}$ is always 1, so

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

3. Expand $\delta(x-t)$ in a Fourier series, over $-\pi < x, t < \pi$.
Solution: It is important to observe that t is within the range of integration, otherwise all of the integrals would vanish due to the properties of the delta function. We begin by finding a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x-t)dx$$

$$= \frac{1}{\pi}.$$

Now, for a_n

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-t)\cos nx dx \\ &= \frac{1}{\pi}\cos nt. \end{aligned}$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-t)\sin nx dx \\ &= \frac{1}{\pi}\sin nt. \end{aligned}$$

The Fourier series is then

$$\begin{aligned} \delta(x-t) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(\cos nt\cos nx + \sin nt\sin nx) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos\left[n(x-t)\right]. \end{aligned}$$

4. Find the Fourier series of

$$f(x) = x, -\pi < x < \pi.$$

Solution: The function is odd, so $a_n = 0$ and $a_0 = 0$, so we only need to evaluate

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx.$$

This can be integrated by parts with

$$\begin{aligned} u &= x & dv &= \sin nx \\ du &= 1 & v &= \frac{-1}{n}\cos nx \end{aligned}$$

giving

$$b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} dx\cos nx\right].$$

The integral is zero by symmetry, so we are left with

$$\begin{aligned} b_n &= \frac{2}{\pi}(-1)\cos n\pi \\ &= \frac{2}{\pi}(-1)^{n+1}. \end{aligned}$$

The Fourier series is therefore

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin nx.$$