Differential Equations

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We've got 4 methods for solving differential equations:

1. Integrating factor.
   For differential equations of the form
   $$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x).$$
2. Variable separable.
3. Homogeneous equations
   This type of differential equation can be written in terms of the variable $\nu = y/x$.
4. Exact differential equations.
   These can be written as $P(x,y)dx + Q(x,y)dy$, where $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Questions

1. (a) Obtain the general solution of
   $$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = (x+1)^6.$$
   Find the specific solution if $y = 2/3$ when $x = 0$.

   (b) Consider the differential equation
   $$x^2\frac{\mathrm{d}y}{\mathrm{d}x} - xy = y^2.$$
   State the order of the equation.
   Find the general solution.

2. Consider the function $f(y) = \tan y$, where y depends upon x.
   Write down the expression for $\frac{\mathrm{d}f}{\mathrm{d}x}$.
   Consider the differential equation
   $$x\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} + \tan y = x^2.$$

Make a substitution of the form $u = g(y)$ to bring this to a form that you know how to solve.
Find the general solution of this differential equation.
Find the particular solution if $y = \pi/4$ when $x = 1$.

3. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^4.$$

Make the substitution $v = 1/y^3$, where v depends upon x.
Show that the general solution is

$$y = \left[\frac{2}{x((Ax)^2 + 3)}\right]^{1/3}.$$

Find the particular solution if $y = (2/3)^{1/3}$ when $x = 1$.

4. Consider the differential

$$(3xy^2 + 2y)dx + (2x^2y + x)dy.$$

Show that this is not exact.
Show that when this is multiplied by x, the differential becomes exact.
Hence, find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3xy^2 + 2y}{2x^2y + x},$$

leaving your solution as an implicit function for $y(x)$.

Solutions

1. (a) Obtain the general solution of

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = (x+1)^6.$$

We divide through by $x+1$, so that the differential equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x+1}y = (x+1)^5.$$

This can then be solved with the integrating factor method, with

$$P(x) = -\frac{4}{x+1} \qquad\qquad Q(x) = (x+1)^5.$$

The integrating factor is

$$\begin{aligned} I &= \exp\left[-4\int \frac{1}{x+1}dx\right] \\ &= \exp\left[-4\ln(x+1)\right] \\ &= (x+1)^{-4}. \end{aligned}$$

The general solution is then

$$\begin{aligned} y(x) &= (x+1)^4\left[\int (x+1) + C\right] \\ &= (x+1)^4 x\left(\frac{x}{2}+1\right) + (x+1)^4 C \end{aligned}$$

Find the specific solution if $y = 2/3$ when $x = 0$.
Plugging in the limits, we find that $C = 2/3$.

(b) Consider the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} - xy = y^2.$$

State the order of the equation.
1st order. Find the general solution.
Looking at the expression, we find it's not exact, integrating factor or separable, leaving homogeneous. Dividing through by xy and re–arranging a bit

$$\begin{aligned} \frac{x}{y}\frac{\mathrm{d}y}{\mathrm{d}x} - 1 &= \frac{y}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} &= \left(\frac{y}{x}\right)^2. \end{aligned}$$

Setting $\nu = y/x$, so that $y = \nu x$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}(\nu x) = \nu + x\frac{\mathrm{d}\nu}{\mathrm{d}x}.$$

This gives

$$\begin{aligned} x\frac{\mathrm{d}\nu}{\mathrm{d}x} + \nu - \nu &= \nu^2 \\ x\frac{\mathrm{d}\nu}{\mathrm{d}x} &= \nu^2. \end{aligned}$$

This brings the equation to a form that is separable

$$\begin{aligned} \int \frac{1}{\nu^2} d\nu &= \int \frac{dx}{x} \\ -\frac{1}{\nu} &= \ln x + C \\ \nu(x) &= -\frac{1}{\ln Ax} \\ y &= -\frac{x}{\ln Ax}. \end{aligned}$$

We re–wrote the constant C as $C = \ln A$ to make the final expression more compact.

2. Consider the function $f(y) = \tan y$, where y depends upon x.
Write down the expression for $\frac{\mathrm{d}f}{\mathrm{d}x}$.
Using the chain rule, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Consider the differential equation

$$x\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} + \tan y = x^2.$$

Make a substitution of the form $u = g(y)$ to bring this to a form that you know how to solve.
As suggested by the first part of the question, we'll try a substitution of the form $u(x) = \tan(y(x))$. This transforms the equation to

$$\begin{aligned} x\frac{\mathrm{d}u}{\mathrm{d}x} + u &= x^2 \\ \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{u}{x} &= x. \end{aligned}$$

Find the general solution of this differential equation.
After the transformation of the previous part, we now have something of the form that can be solved with the integrating factor with

$$P(x) = \frac{1}{x} \qquad\qquad Q(x) = x.$$

The integrating factor

$$
\begin{aligned}
I &= \exp\left[\int \frac{1}{x} dx\right] \\
&= \exp\left[\ln x\right] \\
&= x.
\end{aligned}
$$

The general solution is then

$$
\begin{aligned}
u(x) &= \frac{1}{x}\left[\int x^2 dx + C\right] \\
&= \frac{1}{x}\frac{x^3}{3} + \frac{C}{x} \\
&= \frac{x^2}{3} + \frac{C}{x}.
\end{aligned}
$$

All that remains is to undo the original transformation

$$
\begin{aligned}
y(x) &= \arctan(u(x)) \\
&= \arctan\left[\frac{x^2}{3} + \frac{C}{x}\right].
\end{aligned}
$$

Find the particular solution if $y = \pi/4$ when $x = 1$. Putting in the limits, we have

$$
\frac{\pi}{4} = \arctan\left[C + \frac{1}{3}\right],
$$

where we note that $\arctan 1 = \pi/4$, so that $C + 1/3 = 1$, so $C = 2/3$.

3. Consider the differential equation

$$
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^4.
$$

Make the substitution $v = 1/y^3$, where v depends upon x.
Under this substitution $y = (\nu)^{-1/3}$, so

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{3}\nu^{-4/3}\frac{\mathrm{d}\nu}{\mathrm{d}x} \\
-\frac{1}{3}\frac{\mathrm{d}\nu}{\mathrm{d}x}\nu^{-4/3}\nu^{4/3} + \frac{\nu}{x} &= 1 \\
\frac{\mathrm{d}\nu}{\mathrm{d}x} - \frac{3\nu}{x} &= -3.
\end{aligned}
$$

Show that the general solution is

$$
y = \left[\frac{2}{x((Ax)^2 + 3)}\right]^{1/3}.
$$

After the substitution, we can solve the differential equation with the integrating factor method. The integrating factor is

$$\begin{aligned} I &= \exp\left[-3\int \frac{1}{x}dx\right] \\ &= \frac{1}{x^3}. \end{aligned}$$

The solution is then

$$\begin{aligned} \nu(x) &= x^3\left[-3\int \frac{1}{x^3}dx + C\right] \\ &= \frac{3}{2}X + Cx^3 \\ &= \frac{x}{2}(2Cx^2+3) \\ &= \frac{x}{2}(A^2x^2+3). \end{aligned}$$

We've set $A^2 = 2C$. Undoing the substitution

$$y(x) = \left[\frac{2}{x((Ax)^2+3)}\right]^{1/3}.$$

Find the particular solution if $y = (2/3)^{1/3}$ when $x = 1$.
Putting in the boundary conditions, we find that $A = 0$, so the solution is

$$y(x) = \left(\frac{2}{3x}\right)^{1/3}$$

4. Consider the differential

$$(3xy^2 + 2y)dx + (2x^2y + x)dy.$$

Show that this is not exact.
The differential is of the form $P(x,y)dx + Q(x,y)dy$. An exact differential has the property that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. For this differential

$$\frac{\partial P}{\partial y} = 6xy + 2 \qquad\qquad \frac{\partial Q}{\partial x} = 4xy + 1.$$

These are not equal. Show that when this is multiplied by x, the differential becomes exact.
Multiplying the differential by x gives us

$$(3x^2y^2 + 2xy)dx + (2x^3y + x^2)dy,$$

so that now

$$\frac{\partial P}{\partial y} = 6x^2y + 2x \qquad\qquad \frac{\partial Q}{\partial x} = 6x^2y + 2x.$$

Hence, find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3xy^2 + 2y}{2x^2y + x},$$

leaving your solution as an implicit function for $y(x)$.
As the differential is now exact, so we can solve this in the usual way

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 3x^2y^2 + 2xy \\
F(x, y) &= x^3y^2 + x^2y + g(y) \\
\frac{\partial F}{\partial y} &= 2x^3y + x^2 \\
&= x^3y^2 + x^2y + k(x) \\
F(x, y) &= x^3y^2 + x^2y = \text{const.}
\end{aligned}$$