Partial Differentiation

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1 Partial Differentiation

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \ln(\sin 2x)$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sin 2x} 2\cos 2x = \frac{2}{\tan 2x},\\ y &= \ln u, u = \sin v, v = 2x,\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}.\end{aligned}$$

2. Find $\frac{\mathrm{d}z}{\mathrm{d}t}$ where $z = 2t^2 \sin t$.

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= (2t^2)\cos t + (4t)\sin t\\ &= 2t\left[t\cos t + 2\sin t\right]\end{aligned}$$

3. Find all the partial derivatives of $z = x^2 \sin(2x + 3y)$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= (2)x^2\cos(2x+3y) + \sin(2x+3y)(2x)\\ &= 2x^2\cos(2x+3y) + 2x\sin(2x+3y).\\ \frac{\partial z}{\partial y} &= (3)x^2\sin(2x+3y).\end{aligned}$$

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2 Implicit Differentiation

1. If $x + e^x = 1$, find $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial x}{\partial t}e^x = 1 \\ \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+e^x}. \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + e^x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + e^x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0 \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + e^x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + e^x\left(\frac{1}{1+e^x}\right)^2 = 0 \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{-e^x}{(1+e^x)^3}. \end{aligned}$$

2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ of $xy + 2y - x = 4$.

$$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x+2) = 1 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-y}{x+2}. \end{aligned}$$

3 Some Other Examples

1. If $w = f(ax + by)$ show that

$$b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = 0.$$

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial f}\frac{\partial f}{\partial x} \\ &= \frac{\partial w}{\partial f}a \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial f}\frac{\partial f}{\partial y} \\ &= \frac{\partial w}{\partial f}b \\ \Rightarrow b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} &= ab\frac{\partial w}{\partial f} - ab\frac{\partial w}{\partial f} = 0. \end{aligned}$$

2. If $u = f(x - ct) + g(x - ct)$, show that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial u}{\partial g}\frac{\partial g}{\partial x} \\
&= \frac{\partial u}{\partial f} + \frac{\partial u}{\partial g} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial f^2} + \frac{\partial^2 u}{\partial g^2} \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial f}\frac{\partial f}{\partial t} + \frac{\partial u}{\partial g}\frac{\partial g}{\partial t} \\
&= \frac{\partial u}{\partial f} + \frac{\partial u}{\partial g} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial u}{\partial f}^2 + \frac{\partial u}{\partial g}^2 \\
&= c^2\left(\frac{\partial u}{\partial f} + \frac{\partial u}{\partial g}\right) \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}
\end{aligned}
$$

3. If $z = \cos(xy)$, show that

$$
x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0.
$$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= -y\sin(xy) \\
\frac{\partial z}{\partial y} &= -x\sin(xy) \\
\Rightarrow x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} &= -xy\sin(xy) + xy\sin(xy) = 0.
\end{aligned}
$$

4. The base radius, r, of a cone is decreasing by 0.1 cm/s and the height h is increasing by 0.2 cm/s. Find how the volume $V = \frac{\pi}{3}r^2h$ is changing when $r = 2$ cm and $h = 3$ cm.

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial t} \\
\frac{\partial V}{\partial r} &= \frac{2}{3}\pi r h \\
\frac{\partial V}{\partial h} &= \frac{\pi}{3}r^2 \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \left(\frac{2}{3}\pi r h\right)(-0.1) + \left(\frac{\pi}{3}r^2\right)(0.2) \\
&= -0.42\mathrm{cm}^3\mathrm{s}^{-1}.
\end{aligned}
$$

5. If $z = 2xy - 3x^2y$ and x is increasing by 2 cm/s, find how y must change so that z remains constant when $x = 3$ cm and $y = 1$ cm.

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial x} &= 2y - 6xy \\
\frac{\partial z}{\partial y} &= 2x - 3x^2 \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= (2y - 6xy)\frac{\partial x}{\partial t} + (2x - 3x^2)\frac{\partial y}{\partial t} = 0 \\
&= -31 - 21\frac{\partial y}{\partial t} \\
\Rightarrow \frac{\partial y}{\partial t} &= -\frac{32}{21}\text{cm/s.}
\end{aligned}$$

6. Prove that if $V = \ln\big(x^2 + y^2\big)$ then $\nabla^2 V = 0$.

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{2x}{x^2 + y^2} \\
\frac{\partial V}{\partial y} &= \frac{2y}{x^2 + y^2} \\
\frac{\partial^2 V}{\partial x^2} &= (2x)^2(-1)(x^2 + y^2)^{-2} + 2(x^2 + y^2)^{-1} \\
&= \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2}. \\
\frac{\partial^2 V}{\partial y^2} &= \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} \\
&= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}. \\
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= \frac{1}{(x^2 + y^2)^2}\left[2y^2 - 2x^2 + 2x^2 - 2y^2\right] = 0.
\end{aligned}$$