Physical Applications

James Capers *

December 15, 2021

Abstract

Most of the questions this week are taken from chapter 3 of "Mathematical Methods in the Physical Sciences", by M. L. Boas.

1. Given the curve $y = x^2$ from $x = 0$ to $x = 1$, find

   (a) the area under the curve (that is, the area bounded by the curve, the x axis, and the line $x = 1$;
   We must evaluate the integral

$$A = \int_0^1 dx \int_0^{x^2} dy = \int_0^1 x^2 dx = \frac{1}{3}.$$

   (b) the mass of a plane sheet of material cut in the shape of this area if its density (mass per unit area) is xy;
   To find the mass, we integrate the density over the area.

$$\begin{aligned} M &= \int_0^1 dx \int_0^{x^2} dy xy \\ &= \int_0^1 x dx \int_0^{x^2} y dy \\ &= \int_0^1 x dx \left[\frac{y^2}{2}\right]_0^{x^2} \\ &= \int_0^1 dx \ \frac{x^5}{2} \\ &= \frac{1}{12}. \end{aligned}$$

   (c) the arc length of the curve;
   The element of length is $dl = \sqrt{dx^2 + dy^2}$, which can be re–written

*jrc232@exeter.ac.uk

as $dl = \sqrt{1+(dy/dx)^2}dx$. To find the total length between $x = 0$ and $x = 1$ we need to integrate.

$$L = \int_0^1 \sqrt{1+4x^2}dx.$$

This is a slightly tricky integral, but we can evaluate it with the substitution

$$x = \frac{\sinh u}{2} \qquad\qquad dx = \frac{\cosh u}{2}du.$$

Along the way, we'll need the double angle formulae for the hyperbolic functions, so we'll write these down now

$$\cosh(2x) = 2\cosh^2 x - 1 \qquad\qquad \sinh(2x) = 2\sinh x\cosh x.$$

Putting in the substitution and using the fact that $\cosh^2 x - \sinh^2 x = 1$ and using the double angle formula, we have

$$\begin{aligned} L &= \int \frac{1}{2}\cosh u du\sqrt{1+\sinh^2 u} \\ &= \frac{1}{2}\int du\ \cosh^2 u \\ &= \frac{1}{4}\int du(1+\cosh 2u). \end{aligned}$$

This can be integrated without too much trouble, to give

$$I = \frac{1}{4}\left(u + \frac{1}{2}\sinh 2u\right).$$

Since we didn't bother to change the limits and work out the definite integral in terms of only u, let's undo the substitution. The first term is easy to undo, but for the second term, we re–write $\sinh 2u = 2\sinh u\cosh u$ and then note that $\cosh u = \sqrt{1+\sinh^2 u}$. This gives us

$$I = \frac{1}{4}\sinh^{-1}(2x) + \frac{1}{2}x\sqrt{1+4x^2}.$$

To find L, we must put in the limits. At the lower limit, everything is zero, so putting in the upper limit $x = 1$ gives us

$$L = \frac{1}{4}\sinh^{-1}(2) + \frac{\sqrt{5}}{2} \approx 1.48.$$

(d) the center of mass;
To find the centre of mass, we must evaluate the integrals

$$\bar{x} = \frac{1}{M}\int dA\ x\ \rho \qquad\qquad \bar{y} = \frac{1}{M}\int dA\ y\ \rho.$$

Starting with $\bar{x}$, noting that $1/M = 12$ (from part b), we have

$$\begin{aligned}
\bar{x} &= \frac{1}{M}\int_0^1 dx \int_0^{x^2} dy x^2 y \\
&= 12\int_0^1 dx \left[\frac{x^2y^2}{2}\right]_0^{x^2} \\
&= 12\int_0^1 dx \frac{x^6}{2} \\
&= \frac{12}{14}\left[x^7\right]_0^1 \\
\bar{x} &= \frac{6}{7}.
\end{aligned}$$

And now the $\bar{y}$ integral

$$\begin{aligned}
\bar{y} &= \frac{1}{M}\int_0^1 dx \int_0^{x^2} dy x y^2 \\
&= 12\int_0^1 dx \left[\frac{xy^3}{3}\right]_0^{x^2} \\
&= 12\int_0^1 dx \frac{x^7}{3} \\
&= 4\left[\frac{x^8}{8}\right]_0^1 \\
\bar{y} &= \frac{1}{2}.
\end{aligned}$$

(e) the moments of inertia about the x, y, and z axes of the lamina.
To find the moments of inertia, we need to integral $\ell^2 dM$ over the object, were ℓ is the distance from the axis we are evaluating the moment around. For example, the distance from the x axis is y. Starting with I_x, we have

$$\begin{aligned}
I_x &= \int_0^1 dx \int_0^{x^2} dy (xy)(y^2) \\
&= \int_0^1 dx \left[\frac{xy^4}{4}\right]_0^{x^2} \\
&= \int_0^1 dx \frac{x^9}{4} \\
&= \frac{1}{40}.
\end{aligned}$$

Then, for the moment around the y axis

$$\begin{aligned} I_y &= \int_0^1 dx \int_0^{x^2} dy(xy)(x^2) \\ &= \int_0^1 dx \left[\frac{x^3y^2}{2}\right]_0^{x^2} \\ &= \int_0^1 dx \frac{x^7}{2} \\ &= \frac{1}{16}. \end{aligned}$$

For the moment about the z axis, the integral to evaluate is

$$I_z = \int_0^1 dx \int_0^{x^2} dy(xy)(x^2 + y^2),$$

however we can note that this is just $I_x + I_y = I_z$: this is an example of the parallel axis theorem and means we can just write down $I_z = 1/16 + 1/40 = 7/80$.

2. Given a semi–circular sheet ($x \geq 0$) of material of radius a and constant density ρ, find

   (a) the centroid of the semicircular area;
   By symmetry, we can see that $\bar{y} = 0$ (if you're not convinced, do the integral!). We begin by finding the mass

$$\begin{aligned} M &= \int \rho dA \\ &= \rho \int_0^a r dr \int_{-\pi/2}^{\pi/2} d\theta \\ &= \frac{\rho \pi a^2}{2}. \end{aligned}$$

   Notice that we could have guessed this. A full circle would have mass $\pi a^2 \rho$ so a semi–circle has half of this. Now, to find the x coordinate of the centre of mass, we must evaluate the integral

$$\begin{aligned} \bar{x} &= \frac{1}{M} \int x \rho dA \\ &= \frac{2}{\rho \pi a^2} \rho \int_0^a r^2 dr \int_{-\pi/2}^{\pi/2} \cos\theta d\theta \\ &= \frac{4a}{3\pi}. \end{aligned}$$

   (b) the moment of inertia of the sheet of material about the diameter forming the straight side of the semicircle.

This is the moment of inertia around the y axis, so the distance to this axis is the x coordinate. Written in polar coordinates this is $r \sin \theta$. To find the moment of inertia we evaluate

$$\begin{aligned} I_y &= \int \rho x^2 dA \\ &= \rho \int_0^a r^3 dr \int_{-\pi/2}^{\pi/2} \sin^2 \theta d\theta \\ &= \rho \frac{a^4}{4} \int_{-\pi/2}^{\pi/2} (1 - \cos 2\theta) d\theta \\ &= \rho \frac{a^4 \pi}{8}. \end{aligned}$$

Note that the $\cos 2\theta$ part of the integral does not need to be evaluated as it is an even function over symmetric limits, so the result will be zero.

3. Find the z coordinate of the centroid of a solid cone of height h equal to the radius of the base r and uniform density ρ.
   Also find the moment of inertia of the solid about its axis.

   We begin by finding the mass, working in cylindrical coordinates and taking care that the r integral produces a result that depends upon z, we find that

$$\begin{aligned} M &= \rho \int_0^h dz \int_0^z r dr \int_0^{2\pi} d\theta \\ &= \frac{\rho \pi h^3}{3}. \end{aligned}$$

   To now evaluate the z coordinate of the centre of mass, we evaluate

$$\begin{aligned} \bar{z} &= \frac{1}{M} \rho \int_0^h z dz \int_0^z r dr \int_0^{2\pi} d\theta \\ &= \frac{3}{4} h. \end{aligned}$$

   We can see that both the mass and the position are dimensionally correct.

   To find the moment of inertia, we note that the distance from the z axis is just r, so we just evaluate the integrals as before

$$\begin{aligned} I_z &= \rho \int_0^h dz \int_0^z r^3 dr \int_0^{2\pi} d\theta \\ &= \frac{\rho \pi h^5}{10}. \end{aligned}$$

4. Find the moment of inertia of a solid ball of radius a and constant density ρ about the z axis.

   The mass of a sphere is just $M = (4/3)\pi a^3 \rho$, and the distance from the z axis squared is $x^2 + y^2$. Converting to spherical coordinates gives

$$\begin{aligned} x^2 + y^2 &= r^2 \cos^2 \phi \sin^2 \theta + r^2 \sin^2 \phi \sin^2 \theta \\ &= r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) \\ &= r^2 \sin^2 \theta. \end{aligned}$$

   Remembering that the volume element in spherical coordinates is $dV = r^2 \sin \theta dr d\theta d\phi$, we must evaluate the integral

$$\begin{aligned} I_z &= \rho \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3 \theta d\theta \int_0^a r^4 dr \\ &= \frac{8\pi \rho a^3}{15}. \end{aligned}$$

   We've skipped the steps of the integrals, but everything is pretty standard and the $\sin^3 \theta$ integral can be evaluated noting that $\sin^3 \theta = \sin \theta (1 - \cos^2 \theta)$.