PHY2021 Electromagnetism I
Week 7 Problems: Poisson and Laplace Equation, Magnetic Force, Biot Savart Law

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Question 1 is taken from an exam. Questions 2 and 3 are just to get you used to dealing with the magnetic force. Expressions for surface and volume elements in curvilinear coordinates may be found online or in your notes.

1. (a) Derive Poisson's equation from the differential form of Gauss's law.

   (b) State the condition under which Poisson's equation turns into Laplace's equation.

   (c) A long thin cylindrical conducting shell of radius a has a charge per unit length λ. It is surrounded by a thin conducting cylinder of radius b, such that $b > a$ and the axes of the two cylinders coincide (i.e. they are coaxial). The outer conductor is earthed. By solving Laplace's equation show that the potential at points which are at distance r ($a < r < b$) from the axes of the cylinders can be expressed as

   $$\phi(r) = V\frac{\ln(b/r)}{\ln(b/a)},$$

   where V is the value of the potential at $r = a$. [1]

   (d) Show that

   $$V = \lambda\frac{\ln(b/a)}{2\pi\varepsilon_0}$$

   (e) Derive an expression for the capacitance per unit length of the coaxial cylinders.

[1] Hint: Due to the cylindrical symmetry ϕ satisfies the Laplace equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

in cylindrical coordinates.

2. The Biot-Savart law is

   $$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{d\boldsymbol{l} \times \boldsymbol{r}}{r^3}.$$

   (a) State the meaning of all of the symbols in the equation.

   (b) Using the Biot-Savart law, find the magnetic field a distance z above the centre of a circular loop of wire of radius R, which carries steady current I.

3. You've seen that charged particles in magnetic fields move in helical trajectories, however more exotic paths are followed if the electric and magnetic fields are perpendicular.

Using the Lorentz force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

consider the case where a magnetic field is applied along the x axis, and an electric field points along the z axis. A positive charge is initially at rest at the origin. [2]

[2] Here, we will follow the method of solution given by Griffiths, however a very elegant solution is also presented in "The Classical Theory of Fields" by L. D. Landau and E. M. Lifshitz, however this requires some familiarity with manipulating complex numbers.

(a) Using the fact that $\boldsymbol{F} = m\boldsymbol{a}$, $\boldsymbol{E} = E\hat{\boldsymbol{z}}$, $\boldsymbol{B} = B\hat{\boldsymbol{x}}$ and $\boldsymbol{v} = (\dot{x}\,\hat{\boldsymbol{x}}, \dot{y}\,\hat{\boldsymbol{y}}, \dot{z}\,\hat{\boldsymbol{z}})$, show that the equations of motion are

$$m\ddot{y} = qB\dot{z} \qquad\qquad m\ddot{z} = qE - qB\dot{y}.$$

(b) Identifying the frequency $\omega = qB/m$, write the equations of motion as

$$\ddot{y} = \omega\dot{z} \qquad\qquad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

(c) The general solution to these equations is

$$\begin{aligned} y(t) &= C_1\cos\omega t + C_2\sin\omega t + \left(\frac{E}{B}\right)t + C_3, \\ z(t) &= C_2\cos\omega t - C_1\sin\omega t + C_4, \end{aligned}$$

where C_n are constants of integration.
Use the fact that the particle started at rest ($\dot{y}(0) = 0$, $\dot{z}(0) = 0$) at the origin ($y(0) = 0$, $z(0) = 0$) to show that the trajectory is

$$y(t) = \frac{E}{\omega B}\left(\omega t - \sin\omega t\right), \qquad z(t) = \frac{E}{\omega B}\left(1 - \cos\omega t\right).$$

This path is called a **cycloid**, shown in Figure 1.

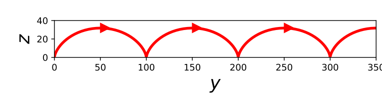

Figure 1: The motion of a charged particle in constant perpendicular electric and magnetic fields. This shape of curve is called a cycloid.