PHY2021 Electromagnetism I
Week 5 & 6 Problems: Electrostatics Review

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These questions are taken from exams. You are given the expressions for $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$ in spherical or cylindrical coordinates if you need them. For these problems, find the results on wikipedia `https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates`.

You are *not* given the divergence theorem, Stokes' theorem or volume and surface elements (you will need to remember these for the exam). If you do not remember these yet, find the results in your notes on online.

1. (a) Starting from the differential form of Gauss law, use the divergence theorem to derive the integral form of Gauss law. Identify all of the symbols you use.

   (b) A sphere of radius R consisting of a dielectric material of permittivity ε carries a free charge density that varies with distance from the centre as

   $$\rho = \rho_0 \left(\frac{r}{R}\right)^3,$$

   where ρ_0 is a constant and r is the distance from the centre of the sphere.
   Derive expressions for the electrostatic field inside and outside the sphere, assuming that the permittivity outside the sphere is ε_0.

   (c) Obtain expressions for the electric displacement and polarisation inside the sphere.

   (d) Obtain an expression for the surface bound charge density at the surface of the sphere.

   (e) Show that the electrostatic potential outside the sphere can be expressed as

   $$\phi(r) = \frac{\rho_0 R^3}{6\varepsilon_0 r}.$$

2. (a) Define the electric dipole moment of a pair of point charges $+q$ and $-q$.

   (b) Show that the electric field $\boldsymbol{E}(\boldsymbol{r})$ at a point $\boldsymbol{r}$ far from an electric dipole with dipole moment $\boldsymbol{p}$ located at the origin can be expressed as

   $$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left[3(\boldsymbol{r}\cdot\boldsymbol{p})\frac{\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3}\right].$$

Explain important steps of the derivation.

(c) Using the above expression, assume that the dipole is aliged along $\hat{\boldsymbol{z}}$. Derive the following expression for the electric field in polar coordinates

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

(d) Give an expression for the dipole moment of a charge distribution $\rho(\boldsymbol{r})$ in a volume V.

(e) Show that a cylinder of radius R and length L centred on the origin, with its axis laying along the z axis, with a volume charge desnity $\rho = \rho_0(z/L)^3$ has a dipole moment with magntiude

$$p = \frac{\rho_0\pi R^2 L^2}{80}.$$

In what direction is the dipole moment pointing?