PHY2021 Electromagnetism I
Week 10 Problems: Maxwell's Equations

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1. (a) Beginning from Ampere's Law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{1}$$

show that

$$-\nabla \cdot \boldsymbol{S} = \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}, \tag{2}$$

where $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ is the Poynting vector.
Identify the meaning of each of the terms.

(b) Using the fact that the energy density of the electromagnetic field is

$$u = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}\right), \tag{3}$$

show that in the absence of currents energy conservation is expressed by

$$\nabla \cdot \boldsymbol{S} + \frac{\partial u}{\partial t} = 0. \tag{4}$$

2. (a) Maxwell's equations in vacuum can be written as [1]

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \boldsymbol{B} = 0, \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} - \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned} \tag{5}$$

Identify the term introduced by Maxwell and, by considering charge conservation, explain why it was necessary to introduce this term.

(b) Using the vector identity $\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$, and considering a region of space where $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$ show that wave equations of the form

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}, \tag{6}$$

can be obtained for *both* $\boldsymbol{E}$ and $\boldsymbol{B}$.

(c) What is the implication of this result?

(d) What is the speed of the waves, in terms of the quantities that appear in Maxwell's equations.

[1] These were first derived by Maxwell around 1861-62, however were only written in their friendly vectorial form by Oliver Heaviside around 1893. Maxwell wrote out all of the components, resulting in 20 equations with 20 unknown variables. Heaviside reduced this to the familiar four equations in four variables that we write today. Despite this, we still call them "Maxwell's equations".