Fourier Transforms

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In the following, we define the Fourier transform and its inverse as

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad (1)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dx. \quad (2)$$

Questions

1. Verify the divergence theorem for the vector field $\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ taking the volume to be a cylinder whose axis is in the z direction and whose base is centred at the origin.

2. Use Stoke's theorem to show that

$$\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = -\pi, \quad (3)$$

   when $\boldsymbol{A} = 3y\hat{\boldsymbol{x}} + 2x\hat{\boldsymbol{y}} - z^3\hat{\boldsymbol{z}}$ and C is the boundary of the surface S, the upper half surface of the sphere: $x^2 + y^2 + z^2 = 1$, $z > 0$.

3. Evaluate $\iint_S (\nabla \times \boldsymbol{a}) \cdot d\boldsymbol{S}$ where $\boldsymbol{a} = (2x - z^2)\hat{\boldsymbol{x}} + (x^3 + yz^3)\hat{\boldsymbol{y}} - x^2y\hat{\boldsymbol{z}}$ and S is the surface of the cone $z = 1 - \sqrt{x^2 - y^2}$ above the $x - y$ plane.

4. Evaluate the Fourier transform of

$$f(x) = \begin{cases} x & \text{for} -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

5. Evaluate the Fourier transform of

$$f(x) = \begin{cases} e^{-\gamma x}\cos(k_0 x) & \text{for } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

6. Evaluate the Fourier transform of

$$f(x) = e^{-x^2/2L^2}.$$

Solutions

1. Verify the divergence theorem for the vector field $\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ taking the volume to be a cylinder whose axis is in the z direction and whose base is centred at the origin.
   Starting with the volume integral, we find that

$$\nabla \cdot \boldsymbol{\rho} = 2$$
$$2\int dV = 2\pi r^2 h.$$

   To evaluate the surface integral, we note that we won't need to integrate over the "caps" because $\boldsymbol{\rho} \cdot \hat{\boldsymbol{z}} = 0$. The surface element for a cylinder is then $d\boldsymbol{S} = r d\theta dz \hat{\boldsymbol{\rho}}$. Putting together the integral, we have

$$\begin{aligned}\iint \boldsymbol{A} \cdot d\boldsymbol{S} &= \iint (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}) \cdot \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{\rho} \rho d\theta dz \\ &= \rho^2 \int_0^{2\pi} d\theta \int_{-h/2}^{h/2} dz \\ &= 2\pi r^2 h.\end{aligned}$$

   So the left hand side of the divergence theorem is equal to the right hand side.

2. Use Stoke's theorem to show that

$$\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = -\pi, \tag{6}$$

   when $\boldsymbol{A} = 3y\hat{\boldsymbol{x}} + 2x\hat{\boldsymbol{y}} - z^3\hat{\boldsymbol{z}}$ and C is the boundary of the surface S, the upper half surface of the sphere: $x^2 + y^2 + z^2 = 1$, $z > 0$.
   One could try to evaluate the line integral directly, but this will be tricky. Instead, we re–write the integral as $\iint (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S}$, using Stoke's theorem. Evaluating the curl gives

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ 3y & 2x & -z^3 \end{vmatrix} = -\hat{\boldsymbol{z}}.$$

   As we are integrating over a sphere, the surface element normal is in the $\hat{\boldsymbol{r}}$ direction. We can note that $-\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = -\cos\theta$, where θ is the usual polar angle from the z axis. As usual, the surface element is $dS = r^2 \sin\theta d\theta d\phi$, so the surface integral is

$$\iint (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} = r^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} (-\cos\theta) \sin\theta d\theta$$
$$= -2\pi \int_0^{\pi/2} \frac{\sin 2\theta}{2} d\theta$$
$$= -\pi \left[-\cos\theta\right]_0^{\pi/2}$$
$$= -\pi(0+1)$$
$$= -\pi.$$

Note that the θ limits are 0 and $\pi/2$ as we are only integrating over a hemisphere.

3. Evaluate $\iint_S (\nabla \times \boldsymbol{a}) \cdot d\boldsymbol{S}$ where $\boldsymbol{a} = (2x - z^2)\hat{\boldsymbol{x}} + (x^3 + yz^3)\hat{\boldsymbol{y}} - x^2 y \hat{\boldsymbol{z}}$ and S is the surface of the cone $z = 1 - \sqrt{x^2 - y^2}$ above the $x - y$ plane.
As with the previous question, the direct integral is hard to evaluate so instead we make use of Stoke's theorem and convert this into a line integral around the circle at the base of the cone in the $x - y$ plane, $\oint \boldsymbol{a} \cdot d\boldsymbol{l}$. In this plane, $z = 0$ and $d\boldsymbol{l} = r d\theta d\hat{\boldsymbol{\theta}}$. Noting that $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}}$, we have

$$\oint \boldsymbol{a} \cdot d\boldsymbol{l} = \oint r d\theta \left[-2x\sin\theta + x^3\cos\theta\right]$$
$$= r^2 \int_0^{2\pi} \left[-2\cos\theta\sin\theta + r^2\cos^4\theta\right] d\theta$$
$$= r^4 \frac{3\pi}{4}$$
$$= \frac{3\pi}{4}$$

4. Evaluate the Fourier transform of

$$f(x) = \begin{cases} x & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

We proceed in the usual way, evaluating the integral

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{-ikx} dx. \tag{8}$$

Integrating by parts $\int v du = uv - \int v du$ with $u = x$ and $dv = e^{-ikx}$, we get $du = 1$ and $v = (-1/ik)e^{-ikx}$, giving

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \left[\frac{ix}{k} e^{-ikx} \Big|_{-1}^{1} - \frac{i}{k} \int_{-1}^{1} e^{-ikx} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{2i \cos k}{k} - \frac{2i}{k^2} \sin k \right] \\ &= i\sqrt{\frac{2}{\pi}} \frac{k \cos k - \sin k}{k^2}. \end{aligned}$$

5. Evaluate the Fourier transform of

$$f(x) = \begin{cases} e^{-\gamma x} \cos(k_0 x) & \text{for } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

To do this, we will write $\cos(k_0 x)$ in exponential form, so that the integral can be performed straight away

$$\begin{aligned} F(k) &= \frac{1}{2\sqrt{2\pi}} \int_0^\infty e^{-x[i(k-k_0)+\gamma]} + e^{-x[i(k+k_0)+\gamma]} \\ &= \frac{1}{2\sqrt{2\pi}} \left[\frac{1}{i(k-k_0)+\gamma} + \frac{1}{i(k+k_0)+\gamma} \right] \\ &= \frac{1}{2\sqrt{2\pi}} \left[\frac{i(k+k_0)+\gamma+i(k-k_0)+\gamma}{\gamma^2 + 2i\gamma k + k_0^2 - k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{ik+\gamma}{(ik+\gamma)^2 + k_0^2}. \end{aligned}$$

6. Evaluate the Fourier transform of

$$f(x) = e^{-x^2/2L^2}.$$

To do this, we need to work out the integral

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2L^2} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2L^2)-ikx} dx. \end{aligned}$$

This is a standard Gaussian integral

$$\int_{-\infty}^{\infty} a^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)+c},$$

with $a = 1/2L^2$, $b = -ik$ and $c = 0$. This gives

$$F(k) = Le^{-k^2L^2/2}.$$

This result can be interpreted as a form of the uncertainty relation in quantum mechanics. The original function $f(x)$ represents a particle localised as a Gaussian in space with $\Delta x \sim L$. The Fourier transform represents the momentum distribution. If we re–write k in terms of the momentum p using $p = \hbar k$, then

$$F(p) = Le^{-p^2 L^2 / 2\hbar^2}.$$

This means that $\Delta p \sim \hbar / L$, so $\Delta p \Delta x \sim L\hbar / L = \hbar$. This is the uncertainty condition, up to numerical factors.