Series

James Capers *

December 15, 2021

Abstract

Most of the questions this week are taken from chapter 5 of "Mathematical Methods for Physicists" by Arfken and Weber.

In the questions below, we'll use the following definitions. The Taylor series is an expansion of a function about a point, a and is defined as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(a)}{4!}(x-a)^4 + \dots \tag{1}$$

The special case of $a = 0$ is called a Maclaurin series.

Binomial series is defined as

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!} x^n. \tag{2}$$

As we will see, this can be derived by applying the Maclaurin series to $(1+x)^m$.

1. Find the Maclaurin series of

$$f(x) = e^x.$$

   The derivative of e^x is e^x and this evaluated at $x = 0$ is 1, making the series easy to compute.

$$\begin{aligned} e^x &= e^0 + xe^0 + \frac{x^2}{2!}e^0 + \frac{x^3}{3!}e^0 + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \end{aligned}$$

2. Find the Maclaurin series of

$$f(x) = \ln(1+x),$$

*jrc232@exeter.ac.uk

and show that for $x = 1$ this becomes the harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}$.
We begin by finding the series expansion.

$$\begin{aligned}
f(0) &= 0 & & \\
f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\
f''(x) &= \frac{11}{(1+x)^2} & f''(0) &= -1 \\
f'''(x) &= \frac{2}{(1+x)^3} & f'''(0) &= 2 \\
f''''(x) &= \frac{-6}{(1+x)^4} & f''''(0) &= -6 \\
f'''''(x) &= \frac{25}{(1+x)^5} & f'''''(0) &= 24
\end{aligned}$$

so that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

Then, for $x = 1$, this becomes

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$$

which can be written as $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}$.

3. The total relativistic energy of a particle of mass m and velocity v is

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

Compare this with the classical kinetic energy $mv^2/2$.
To make the comparison, let's say that $x = v^2/c^2$ and expand $(1-x)^{-1/2}$ for $x = 0$. This corresponds to the limit where $v \ll c$, where we expect the effects of relativity to be small. Making the expansion

$$\begin{aligned}
f(0) &= 1 & & \\
f'(x) &= \frac{1}{2}(1-x)^{-3/2} & f'(0) &= \frac{1}{2} \\
f''(x) &= \frac{3}{4}(1-x)^{-5/2} & f''(0) &= \frac{3}{4} \\
f'''(x) &= \frac{15}{8}(1-x)^{-7/2} & f'''(0) &= \frac{15}{8}
\end{aligned}$$

so that the energy expansion, in terms of v^2/c^2 is

$$E = mc^2 \left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \frac{15}{8}\frac{1}{3!}\left(\frac{v^2}{c^2}\right)^3 + \dots\right]$$
$$= mc^2 + \frac{1}{2}mv^2 + \dots$$

The first term is the mass–energy and the second term is the usual classical kinetic energy. All of the higher terms represent relativistic corrections to the energy as $v \to c$.

4. By applying the Maclaurin series to $(1+x)^m$, derive the Binomial series. Evaluating the series, we have

$$\begin{aligned} f(0) &= 1 & & \\ f'(x) &= m(1+x)^{m-1} & f'(0) &= m \\ f''(x) &= m(m-1)(1+x)^{m-2} & f''(0) &= m(m-1) \\ f'''(x) &= m(m-1)(m-2)(1+x)^{m-3} & f'''(0) &= m(m-1)(m-2). \end{aligned}$$

Putting this together, we have

$$(1+x)^m = 1 + mx + m(m-1)\frac{x^2}{2} + m(m-1)(m-2)\frac{x^3}{3!} + \dots$$

Comparing this with the binomial series, which is usually written as

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k, \quad \binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$$

we can see that the two are identical.

5. Derive the geometric series by expanding

$$f(x) = \frac{1}{1-x}.$$

around $x = 0$.
Evaluating the series, we have

$$\begin{aligned} f(0) &= 1 & & \\ f'(0) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\ f''(0) &= \frac{2}{(1-x)^3} & f''(0) &= 2 \\ f'''(0) &= \frac{6}{(1-x)^4} & f'''(0) &= 6 \\ f''''(0) &= \frac{24}{(1-x)^5} & f''''(0) &= 24 \end{aligned}$$

so that

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

This is the geometric series and clearly only converges when $|x| < 1$. As long as this condition is met, each term is smaller than the one before it and the sum converges.

6. (a) Given that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

show that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

We begin by noting that $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. Now, since we know the expansion of $\ln(1+x)$, we can find the expansion for $\ln(1-x)$ without any differentiation, by just replacing $x \to -x$ in the expansion. This gives us

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

so that

$$\begin{aligned} \ln\left(\frac{1+x}{1-x}\right) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right) \\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right). \end{aligned}$$

(b) Expand $f(x) = \arctan x$ around $x = 0$.
Evaluating the series, we have

$$\begin{aligned} f(0) &= 1 \\ f'(x) &= (1+x^2)^{-1} & f'(0) &= 1 \\ f''(x) &= -2x(1+x^2)^{-2} & f''(0) &= 0 \\ f'''(x) &= 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2} & f'''(0) &= -2 \\ f''''(x) &= -48x^3(1+x^2)^{-4} + 24x(1+x^2)^{-2} & f''''(0) &= 0 \\ f'''''(x) &= 384x^4(1+x^2)^{-5} - 288x^2(1+x^2)^{-4} + 24(1+x^2)^{-3} & f'''''(0) &= 24. \end{aligned}$$

Giving

$$\arctan(x) = 1 - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

(c) Expand using the binomial theorem

$$f(t) = \frac{1}{1+t^2}.$$

Expanding using the binomial theorem we wrote down in question 4, replacing x with t^2, we get

$$(1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + t^8 + \dots$$

(d) Using this expansion, integrate term by term to show that

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Integrating term by term, we get

$$\begin{aligned}\int_0^x \frac{dt}{1+t^2} &= \int_0^x dt - \int_0^x dt\ t^2 + \int_0^x dt\ t^4 + \dots \\ &= [t]_0^x - \left[\frac{t^3}{3}\right]_0^x + \left[\frac{t^5}{5}\right]_0^x + \dots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \\ &= \arctan(x).\end{aligned}$$

(e) By comparing the series, show that

$$\arctan x = \frac{i}{2} \ln\left(\frac{1-ix}{1+ix}\right).$$

We know from part (a) that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

so using log laws we have

$$\ln\left(\frac{1-x}{1+x}\right) = -2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

Now, we know that

$$\begin{aligned}\frac{i}{2}\ln\left(\frac{1-ix}{1+ix}\right) &= -2\frac{i}{2}\left(ix + \frac{i^3x^3}{3} + \frac{i^5x^5}{5} + \dots\right) \\ &= \frac{1}{i}\left(ix - i\frac{x^3}{3} + i\frac{x^5}{5} + \dots\right) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \\ &= \arctan(x).\end{aligned}$$

We made use of the fact that $-i = 1/i$.