Applications of Integration

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November 17, 2021

1 Surfaces, Areas and Volumes

1. Find the length of the curve $y = \frac{x}{2} - \frac{x^2}{4} + \frac{1}{2}\ln(1-x)$ between $x = 0$ and $x = 1/2$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2} - \frac{x}{2} - \frac{1}{2(x-1)} \\
&= \frac{1}{2}\left(1 - x - \frac{1}{x-1}\right) \\
&= \frac{(1-x)^2 - 1}{2(1-x)} \\
\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= \frac{(1-x)^4 - 2(1-x)^2 + 1}{4(1-x)^2} \\
l &= \int_0^{1/2} \sqrt{1 + \frac{(1-x)^4 - 2(1-x)^2 + 1}{4(1-x)^2}} dx \\
&= \int_0^{1/2} \sqrt{\frac{4(1-x)^2 + (1-x)^4 - 2(1-x)^2 + 1}{4(1-x)^2}} dx \\
&= \int_0^{1/2} \frac{\sqrt{\left[(1-x)^2 + 1\right]^2}}{2(1-x)} dx \\
&= \int_0^{1/2} \frac{(1-x)^2 + 1}{2(1-x)} dx \\
&= \frac{1}{2}\int_0^{1/2} \left(1 - x + \frac{1}{1-x}\right) dx \\
&= \frac{1}{2}\left[x - \frac{x^2}{2} - \ln(1-x)\right]_0^{1/2} \\
l &= \frac{3}{16} + \frac{1}{2}\ln 2
\end{aligned}$$

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2. The parametric equations of a curve are $x = e^t \sin t$,$y = e^t \cos t$. If the arc of this curve, between $t = 0$ and $t = \frac{\pi}{2}$, rotates through a complete revolution about the x-axis, calculate the area of the surface generated.

$$A = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, dt$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t(\sin t + \cos t) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = e^t(\cos t - \sin t)$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 2e^{2t}$$

$$\begin{aligned} A &= \int_0^{\pi/2} 2\pi e^t \cos t \sqrt{2} e^t\, dt \\ &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \cos t dt. \end{aligned}$$

This can now be integrated by parts, with $u = e^{2t}$, $du = 2e^{2t}$, $dv = \cos t$ and $v = \sin t$.

$$\begin{aligned} I &= \int_0^{\pi/2} e^{2t} \cos t dt \\ &= \left[e^{2t} \sin t\right]_0^{\pi/2} - 2 \int_0^{\pi/2} e^{2t} \sin t dt \\ &= e^\pi - 2 \int_0^{\pi/2} e^{2t} \sin t dt. \end{aligned}$$

We need to integrate by parts again, this time with $u = e^{2t}$, $du = 2e^{2t}$, $dv = \sin t$, $v = -\cos t$.

$$\begin{aligned} I &= e^\pi + 2\left[e^{2t} \cos t\right]_0^{\pi/2} - 4 \int_0^{\pi/2} e^{2t} \cos t dt \\ &= e^\pi - 2 - 4I \\ I &= \frac{e^\pi - 2}{5}. \end{aligned}$$

We then have the final answer

$$A = \frac{2\sqrt{2}\pi(e^\pi - 2)}{5}.$$

3. The line $y = 2x$ and the parabola $y^2 = 16x$ intersect at $x = 4$. Find, by double integral, the area enclosed by $y = 2x$, $y^2 = 16x$, $x = 1$ and the

point of intersection $x = 4$.

$$
\begin{aligned}
A &= \int_0^4 dx \int_{2x}^{4\sqrt{x}} dy \\
&= \int_0^4 dx \left(4\sqrt{x} - 2x\right) \\
&= \left[\frac{8}{3}x^{3/2} - x^2\right]_0^4 \\
&= \frac{11}{3}.
\end{aligned}
$$

4. Determine the area bounded by the curves $x = y^2$ and $x = 2y - y^2$. The first step is to find the point of intersection. By equating the two expressions $y^2 = 2y - y^2$ we find that the intersection points are at $y = 0$ and $y = 1$. Now, to find the area

$$
\begin{aligned}
A &= \int_0^1 dy \int_{y^2}^{2y-y^2} dx \\
&= \int_0^1 dy 2y - y^2 \\
&= \left[y^2 - \frac{2}{3}y^3\right]_0^1 \\
&= \frac{1}{3}.
\end{aligned}
$$

5. A rectangular block is bounded by the co-ordinate planes of reference and the planes $x = 3,\ y = 4,\ z = 2$. Its density at any point is numerically equal to the square of its distance from the origin. Find the total mass of the solid.

The density at any point is $\rho = x^2 + y^2 + z^2$ so the total mass is

$$\begin{aligned}
m &= \int \rho dV \\
&= \int_0^3 dx \int_0^4 dy \int_0^2 dz(x^2 + y^2 + z^2) \\
&= \int_0^2 \int_0^4 dydz \left[\frac{x^3}{3} xy^2 + xz^2\right]_0^3 \\
&= \int_0^2 \int_0^4 dydz(9 + 3y^2 + 3z^2) \\
&= \int_0^2 dz \left[9y + y^3 + 3yz^2\right]_0^4 \\
&= \int_0^2 dz(100 + 12z^2) \\
&= \left[100z + 4z^3\right]_0^2 \\
&= 232.
\end{aligned}$$

2 Multiple Integrals

1. Evaluate
$$I = \int_0^a dx \int_0^{y_i} dy(x - y),$$
where $y_i = \sqrt{a^2 - x^2}$.
$$\begin{aligned}
I &= \int_0^a \left[xy - \frac{y^2}{2}\right]_0^{y_i} \\
&= \int_0^a x\sqrt{a^2 - x^2} - \frac{a^2 - x^2}{2} dx \\
&= \frac{1}{2}\left[-\frac{2}{3}(a^2 - x^2)^{3/2} - a^2 x + \frac{x^3}{3}\right]_0^a \\
&= \frac{1}{2}\left(\frac{2a^3}{3} - a^3 + \frac{a^3}{3}\right) = 0.
\end{aligned}$$

2. Evaluate
$$I = \int_0^a \int_0^b \int_0^c (x^2 + y^2) dxdydz.$$

$$
\begin{aligned}
I &= \int_0^a \int_0^b \left[\frac{x^3}{3} + xy^2\right]_0^c dydz \\
&= \int_0^a \int_0^b \left(\frac{c^3}{3} + cy^2\right) dydz \\
&= \int_0^a \left[\frac{c^3 y}{3} + \frac{cy^3}{3}\right]_0^b dz \\
&= \int_0^a \left(\frac{c^3 b}{3} + \frac{cb^3}{3}\right) dz \\
&= \frac{abc}{3}(c^2 + b^2).
\end{aligned}
$$

3. Evaluate

$$
I = \int_0^\pi \int_0^{\pi/2} \int_0^r x^2 \sin\theta dx d\theta d\phi.
$$

$$
\begin{aligned}
I &= \int_0^\pi \int_0^{\pi/2} \left[\frac{x^3}{3}\sin\theta\right]_0^r d\theta d\phi \\
&= \int_0^\pi \int_0^{\pi/2} \left(\frac{r^3}{3}\sin\theta\right) d\theta d\phi \\
&= \int_0^\pi \left[-\frac{r^3}{3}\cos\theta\right]_0^{\pi/2} d\phi \\
&= \int_0^\pi \frac{r^3}{3} d\phi \\
&= \frac{\pi r^3}{3}.
\end{aligned}
$$