PHY2021 Electromagnetism I
Week 3 Problems: Gauss Law

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The divergence theorem states

$$\iiint \nabla \cdot \boldsymbol{A} dV = \iint_{\partial S} \boldsymbol{A} \cdot \hat{\boldsymbol{n}} dS,$$

where $\boldsymbol{A}$ is a vector field, dV is a volume enclosed by the surface dS and $\hat{\boldsymbol{n}}$ is a unit vector normal to the surface ∂S and points outwards.

Stoke's theorem states that

$$\oint \boldsymbol{A} \cdot d\boldsymbol{l} = \iint_{\partial S} \nabla \times \boldsymbol{A} \cdot \hat{\boldsymbol{n}} dS,$$

where $\boldsymbol{A}$ is a vector field, $d\boldsymbol{l}$ is a closed loop which encloses a surface ∂S. Again, the vector $\hat{\boldsymbol{n}}$ is normal to the surface, in the positive sense.

1. Verify the divergence theorem for the vector field $\hat{\boldsymbol{r}}$ and the hemispherical surface bounded by the $x - y$ plane and the surface $x^2 + y^2 + z^2 = 16$ with $z > 0$.

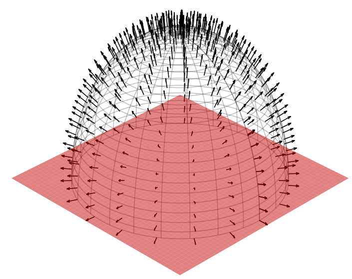

Figure 1: Schematic setup of the situation described in question 1. A hemisphere of radius 4 is bounded by the $z = 0$ plane (red). Think about how this will change the limit of integration over ϕ (will it still be 0 to π?).

2. Evaluate

$$\oint \boldsymbol{a} \cdot d\boldsymbol{l}$$

around the circle $x^2 + y^2 = b^2$ for

$$\boldsymbol{a} = \frac{\boldsymbol{r}}{r^4}$$

both directly and using Stoke's theorem.
Hint: Around the circle $d\boldsymbol{l} = rd\theta\hat{\boldsymbol{\theta}}$. The line integral is most easily evaluated using cylindrical coordinates.
Is the field conservative?

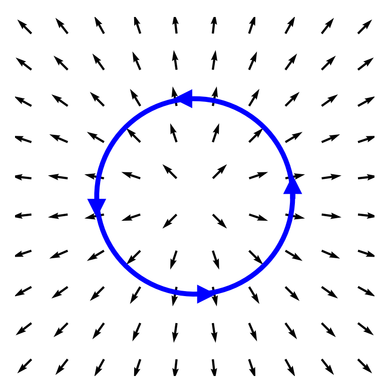

Figure 2: The situation described in question 2. A closed loop (shown in blue) must be integrated around. This can then be competed to the surface integral of the curl of the vector field.

3. (a) By considering the flux through a closed surface around a point charge, derive the integral form of Gauss theorem. The electric field produced by a point charge q is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{r}}}{r^2}.$$

   (b) Use the divergence theorem to convert this to the differential form of Gauss theorem.

4. Use Gauss theorem to calculate the electric field outside a uniformly charged sphere of radius R and total charge Q.